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Homogeneous structures on three-dimensional Lorentzian manifolds

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Abstract

We prove that any non-symmetric three-dimensional homogeneous Lorentzian manifold is isometric to a Lie group equipped with a left-invariant Lorentzian metric. We then classify all three-dimensional homogeneous Lorentzian manifolds.

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1. Introduction

A pseudo-Riemannian manifold (M, g) is *homogeneous* provided that, for any points $p, q \in M$, there exists an isometry ϕ such that $\phi(p) = q$; it is *locally homogeneous* if there is a local isometry mapping a neighborhood of p into a neighborhood of q [12]. We recall here a few examples of results concerning homogeneous and locally homogeneous manifolds, in the Riemannian and pseudo-Riemannian case (in particular, in Lorentzian geometry).

Gadea and Oubiña [9] introduced the notion of *homogeneous pseudo-Riemannian structure*, in order to obtain a characterization of reductive homogeneous pseudo-Riemannian manifolds, similar to the one given for homogeneous Riemannian manifolds by Ambrose and Singer [1] (see also [19]).

A pseudo-Riemannian manifold (M, g) is *curvature homogeneous up to order k* if, for any points $p, q \in M$, there exists a linear isometry $\phi : T_pM \to T_qM$ such that $\phi * (\nabla^i R(q)) = \nabla^i R(p)$ for all $i \le k$. A locally homogeneous space is curvature homogeneous of any order k. Conversely, if k is sufficiently high, curvature homogeneity up to order k implies local homogeneity. This result was proved by Singer [17] for Riemannian manifolds. Through the equivalence theorem for G-structures due to Cartan and Sternberg [18], Singer's result can be extended to the pseudo-Riemannian case.

Given a pseudo-Riemannian manifold (M, g), its *Singer index* k_M is the smallest integer such that curvature homogeneity up to order $k > k_M$ implies local homogeneity. Singer's construction [17] shows that $k_M \le (1/2)n(n-1)$,

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where $n = \dim M$. For Riemannian manifolds, this upper bound was improved by Gromov [10], who proved that $k_M \leq (3/2)n - 1$.

In some cases, these estimates can be further improved. For example, if $\dim M = 2$, then curvature homogeneity (up to order 0) already implies local homogeneity. In [16], Sekigawa proved that a three-dimensional Riemannian manifold, which is curvature homogeneous up to order one, is locally homogeneous. Curvature homogeneous Lorentzian spaces have been investigated in several papers (see for example [2–4,6,7]). In particular, Bueken and Vanhecke [4] gave examples of three-dimensional Lorentzian manifolds which are curvature homogeneous up to order one but not locally homogeneous. In [3], Bueken and Djorić determined all three-dimensional Lorentzian manifolds which are curvature homogeneous up to order one, and also showed that curvature homogeneity up to order two is sufficient for a three-dimensional Lorentzian manifold to be homogeneous.

In [16], Sekigawa also proved that a three-dimensional connected, simply connected and complete homogeneous Riemannian manifold is either symmetric or it is a Lie group endowed of a left-invariant Riemannian metric. Taking into account the classification of three-dimensional Riemannian Lie groups given by Milnor [11], this result permits one to determine all three-dimensional homogeneous Riemannian manifolds.

To our knowledge, while several interesting examples of three-dimensional homogeneous Lorentzian manifolds are known [3,8,14,15], a complete classification result has not been given yet. The main purpose of this paper is to prove the following

Theorem 1.1. *Let (M, g) be a three-dimensional connected, simply connected, complete homogeneous Lorentzian manifold. Then, either (M, g) is symmetric, or it is isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric.*

Theorem 1.1, together with the results on three-dimensional Lorentzian Lie groups obtained by Cordero and Parker [8] and Rahmani [15], leads to the classification of three-dimensional homogeneous Lorentzian manifolds.

The paper is organized in the following way. Section 2 will be devoted to recalling some basic facts and results about homogeneous pseudo-Riemannian structures. In Section 3 we shall prove Theorem 1.1. The classification of three-dimensional homogeneous Lorentzian manifolds will be given in Section 4. In Section 5, we shall complete the description of three-dimensional homogeneous Lorentzian manifolds, by classifying three-dimensional Lorentzian symmetric spaces.

2. Preliminaries

Let M be a connected manifold and g a pseudo-Riemannian metric of signature (m, n) on M. We denote by ∇ the Levi-Civita connection of (M, g) and by R its curvature tensor. The following definition was introduced by Gadea and Oubiña:

Definition 2.1 (*[9]*). A *homogeneous pseudo-Riemannian structure* on (M, g) is a tensor field T of type $(1, 2)$ on M, such that the connection $\tilde{\nabla} = \nabla - T$ satisfies

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla} R = 0, \qquad \tilde{\nabla} T = 0. \tag{2.1}$$

The geometric meaning of the existence of a homogeneous pseudo-Riemannian structure is explained by the following

Theorem 2.2 (*[9]*). *Let (M, g) be a connected, simply connected and complete pseudo-Riemannian manifold. Then, (Mg) admits a pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.*

It must be noted that any homogeneous Riemannian manifold is reductive, while a homogeneous pseudo-Riemannian manifold need not be reductive. We now recall briefly the essential steps of the proof of Theorem 2.2, referring the reader to [9] for further details.

Assume first that $(M = G/H, g)$ is a homogeneous reductive pseudo-Riemannian manifold, G and H being a group of isometries acting transitively and effectively on (M, g) and the isotropy group at an arbitrary point $p \in M$, respectively. Let α belong to the Lie algebra $\mathfrak{g}$ of G and α^* be the vector field on M generated by the one-parameter group of isometries $\{\exp(t\alpha) : t \in \mathbb{R}\}$. The Lie algebra of the isotropy group H is $\mathfrak{h} = \{\alpha \in \mathfrak{g} : \alpha^*_p = 0\}$. As is well known, $M = G/H$ being reductive means that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{m}$ is stable under $\mathrm{Ad}(H)$.

The *canonical connection* $\tilde{\nabla}$ associated with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is uniquely determined by

$$(\tilde{\nabla}_{\alpha^*}\beta^*)_p = [\alpha^*, \beta^*]_p = -[\alpha, \beta]^*_p,$$

for all $\alpha, \beta \in \mathfrak{g}$.

Then, the difference tensor field $T = \nabla - \tilde{\nabla}$ is a homogeneous pseudo-Riemannian structure on (M, g).

Conversely, the existence of a pseudo-Riemannian homogeneous structure T on (M, g) leads to the existence of a connection $\tilde{\nabla} = \nabla - T$ on M, which is complete and ensures the existence, given two points $p, q \in M$, of a global isometry mapping p to q. Then, there exists a group G of isometries acting transitively on M, such that $M = G/H$ is reductive, and $\tilde{\nabla}$ is the canonical connection associated with this reductive decomposition.

Note that two different homogeneous structures T_1 and T_2 on a pseudo-Riemannian homogeneous manifold (M, g), can give rise to the same Lie algebra $\mathfrak{g}$ with different decompositions: $\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{h}_1 = \mathfrak{m}_2 \oplus \mathfrak{h}_2$. Different homogeneous structures T_1 and T_2 on (M, g) can also give non-isomorphic Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ [19, p. 36].

We want to emphasize here the special case when for all $\alpha, \beta \in \mathfrak{g}$, we have $\tilde{\nabla}_{\alpha^*}\beta^* = 0$ or, in other words, $T_{\alpha^*}\beta^* = \nabla_{\alpha^*}\beta^*$. Let A denote the *Kostant operator*, defined, for any tangent vector field X on M, by

$$A_X Y = -\nabla_Y X.$$

If $\alpha^*_p = 0$ and $(A_{\alpha^*})_p = 0$, then $\alpha = 0$, since the representation ρ of $\mathfrak{h}$ in T_pM, defined by $\rho(\alpha) = -(A_{\alpha^*})_p$, is faithful [9, p. 452].

Assume now that $\tilde{\nabla}_{\alpha^*}\beta^* = 0$ for all $\alpha, \beta \in \mathfrak{g}$ and consider $\alpha \in \mathfrak{h}$. Then, by definition, $\alpha^*_p = 0$. Moreover, we have

$$0 = (\tilde{\nabla}_{\alpha^*}\beta^*)_p = [\alpha^*, \beta^*]_p,$$

for all $\beta \in \mathfrak{g}$. Since ∇ is the Levi-Civita connection of M and $\alpha^*_p = 0$, we then have

$$0 = [\alpha^*, \beta^*]_p = \nabla_{\alpha^*_p}\beta^* - \nabla_{\beta^*_p}\alpha^* = -\nabla_{\beta^*_p}\alpha^* = (A_{\alpha^*})_p\beta^*.$$

Consider $Y \in T_pM$. Since G acts transitively on M, there exists $\beta \in \mathfrak{g}$ such that $\beta^*_p = Y$. Therefore, $(A_{\alpha^*})_pY = (A_{\alpha^*})_p\beta^* = 0$, for any Y, that is, $(A_{\alpha^*})_p = 0$ and so, $\alpha = 0$.

Thus, when $\tilde{\nabla}_{\alpha^*}\beta^* = 0$, we have that $\mathfrak{h} = 0$ and so, M itself carries a Lie group structure, unique up to isomorphisms. In this way, we proved the following

Lemma 2.3. *Let (M, g) be a connected, simply connected and complete pseudo-Riemannian manifold. If M admits a homogeneous pseudo-Riemannian structure T such that $T_XY = \nabla_XY$ for all X, Y vector fields tangent to M, then M has a Lie group structure, unique up to isomorphisms, and g is left-invariant.*

3. Three-dimensional homogeneous Lorentzian structures

Let (M, g) be a connected three-dimensional Lorentzian manifold. Its curvature tensor is completely determined by the Ricci tensor ϱ, defined, for any point $p \in M$ and $X, Y \in T_pM$, by

$$\varrho(X, Y)_p = \sum_{i=1}^{3} \varepsilon_i g(R(X, e_i)Y, e_i),$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of T_pM and $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all i. Throughout the paper, if not stated otherwise, we shall assume that e_3 is *timelike*, that is, $g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = 1$. Because of the symmetries of the curvature tensor, the Ricci tensor ϱ is symmetric [12]. So, the *Ricci operator* Q, defined by $g(QX, Y) = \varrho(X, Y)$, is self-adjoint. In the Riemannian case, there always exists an orthonormal basis diagonalizing Q, while in the Lorentzian case four different cases can occur ([12, p. 261], [3]), and there exists a pseudo-orthogonal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, such that Q takes one of the following forms, called *Segre types*:

$$\text{Segre type } \{11, 1\} : \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \text{Segre type } \{1z\bar{z}\} : \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \tag{3.1}$$

$$\text{Segre type } \{21\}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & \varepsilon \\ 0 & -\varepsilon & b-2\varepsilon \end{pmatrix}, \qquad \text{Segre type } \{3\}: \begin{pmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{pmatrix}.$$

When (M, g) is curvature homogeneous (up to order zero), Q has the same Segre type at any point $p \in M$ and has constant eigenvalues. Moreover, if (M, g) is curvature homogeneous up to order two, starting from a pseudo-orthonormal basis $\{(e_i)_p\}$ at a fixed point p, we can use the linear isometries from T_pM into the tangent spaces at any other point, to construct a pseudo-orthonormal frame field $\{e_i\}$, such that the components of ϱ, $\nabla\varrho$ and $\nabla^2\varrho$ with respect to $\{e_i\}$ remain constant along M. With respect to such a frame field $\{e_i\}$, we now put

$$\nabla_{e_i} e_j = \sum_k \varepsilon_j B_{ijk} e_k. \tag{3.2}$$

Clearly, the functions B_{ijk} determine completely the Levi-Civita connection, and conversely. Note that from $\nabla g = 0$ it follows at once that

$$B_{ikj} = -B_{ijk}, \tag{3.3}$$

for all i, j, k. In particular,

$$B_{ijj} = 0 \tag{3.4}$$

for all indices i and j. We are now ready to give the

Proof of Theorem 1.1. Let (M, g) be a connected, simply connected and complete three-dimensional homogeneous Lorentzian manifold, and $\{e_i\}$ a global pseudo-orthonormal frame field on M, such that the components of ϱ, $\nabla\varrho$ and $\nabla^2\varrho$ with respect to $\{e_i\}$ are globally constant. Easy calculations show that

$$\nabla_i \varrho_{jk} = -\sum_t (\varepsilon_j B_{ijt} \varrho_{tk} + \varepsilon_k B_{ikt} \varrho_{tj}) \tag{3.5}$$

and

$$\nabla^2_{ri} \varrho_{jk} = -\sum_t (\varepsilon_i B_{rit} \nabla_t \varrho_{jk} + \varepsilon_j B_{rjt} \nabla_i \varrho_{tk} + \varepsilon_k B_{rkt} \nabla_i \varrho_{tj}), \tag{3.6}$$

for all indices i, j, k, r. We want to prove that whenever (M, g) is not symmetric, there exists a homogeneous Lorentzian structure T such that $T_X Y = \nabla_X Y$ for all X, Y vector fields tangent to M. Then, by Lemma 2.3, M is a Lie group and g a left-invariant Lorentzian metric. To prove the existence of such a tensor T, it is enough to show that, with respect to a suitable pseudo-orthonormal frame field $\{e_i\}$, all B_{ijk} are constants. Then, we can define T by

$$T_{e_i} := \frac{1}{2} \sum_{jk} B_{ijk} e_j \wedge e_k, \tag{3.7}$$

for all i, where $e_j \wedge e_k(X) = g(e_j, X)e_k - g(e_k, X)e_j$. From (3.7) it follows at once that $T_{e_i} e_j = \nabla_{e_i} e_j$ for all i, j. So, $\tilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j - T_{e_i} e_j = 0$ for all i, j. Moreover, $\tilde{\nabla}$ satisfies conditions (2.1) of Definition 2.1, and we can apply Lemma 2.3 to conclude that M is a Lie group and g a left-invariant Lorentzian metric.

Note that the constancy of all B_{ijk} is equivalent to the constancy of all $g([e_i, e_j], e_k)$, since the well known *Koszul formula* yields

$$2\varepsilon_j \varepsilon_k B_{ijk} = 2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) - g([e_j, e_k], e_i) + g([e_k, e_i], e_j). \tag{3.8}$$

We shall treat separately different cases, according with the Segre type of the Ricci operator of (M, g).

(I) *Segre type* $\{11, 1\}$.

In this case, Q is diagonal, that is, $\varrho_{ij} = \varepsilon_i \delta_{ij} q_i$, for all i, j, where $q_1 = a$, $q_2 = b$ and $q_3 = c$ denote the eigenvalues of Q. Hence, (3.5) simplifies as follows:

$$\nabla_i \varrho_{jk} = -\varepsilon_j \varepsilon_k (q_j - q_k) B_{ijk}. \tag{3.9}$$

In particular, from (3.9) we get that $\nabla_i \varrho_{jj} = 0$ for all i, j. When $a = b = c$, (M, g) is an *Einstein manifold* and so, being three-dimensional, it has constant sectional curvature. In particular, it is symmetric. Then, we are left with the following cases:

I(a) $a \neq b \neq c \neq a$. In this case, $q_j - q_k$ is different from zero for all $j \neq k$ and so, by (3.9) it follows at once that B_{ijk} is constant for all $j \neq k$. Taking into account (3.4), all B_{ijk} are then constant.

I(b) $a = b \neq c$. Writing (3.9) with $(j,k) = (1,2)$ we get $\nabla_i \varrho_{12} = 0$, while for $(j,k) = (1,3)$ and $(j,k) = (2,3)$ we respectively obtain that B_{i13} and B_{i23} are constant for all i. We shall prove that, unless (M,g) is symmetric, there exists a suitable pseudo-orthonormal frame field $\{e_i\}$ with respect to which also the B_{i12} are constant for all i.

(M,g) being homogeneous, the *scalar curvature* τ is constant. Hence, from the well known formula $d\tau = 2\mathrm{div}\varrho$ [12, p. 88], we get

$$0 = e_i(\tau) = 2\sum_j \nabla_j \varrho_{ij}, \tag{3.10}$$

for all $i = 1,2,3$. Writing (3.10) for $i = 1,2,3$, we get

$$\nabla_1\varrho_{13} + \nabla_2\varrho_{23} = \nabla_3\varrho_{13} = \nabla_3\varrho_{23} = 0,$$

that is, by (3.9),

$$B_{113} + B_{223} = B_{313} = B_{323} = 0. \tag{3.11}$$

Since $B_{313} = B_{323} = 0$, the integral curves of e_3 are geodesic. Therefore, we can choose $\{e_i\}$ so that $\nabla_{e_3} e_i = 0$, that is, $B_{3ij} = 0$ for all i, j. Since e_1, e_2 are spacelike, the rest of this case can be treated exactly like the corresponding Riemannian case in [16]. We report here these arguments, referring the reader to [16] for more details.

We write (3.6) for $(i,j,k) = (1,2,3)$ and for $(i,j,k) = (2,2,3)$. Taking into account (3.11), we get

$$\begin{cases} (\nabla_1\varrho_{13} - \nabla_2\varrho_{23})B_{r12} = \nabla^2_{r1}\varrho_{23} + \nabla_3\varrho_{23}B_{r13}, \\ (\nabla_1\varrho_{23} + \nabla_2\varrho_{13})B_{r12} = \nabla^2_{r2}\varrho_{23} + \nabla_3\varrho_{23}B_{r23}. \end{cases} \tag{3.12}$$

Since all the components of ϱ and $\nabla\varrho$ are constant, from (3.12) it follows that the B_{r12} are constant, unless $\nabla_1\varrho_{13} = \nabla_2\varrho_{23}$ and $\nabla_1\varrho_{23} = -\nabla_2\varrho_{13}$. In the last case, from (3.9) and (3.11) we then get $B_{113} = B_{223} = 0$ (and so, $\nabla_1\varrho_{13} = \nabla_2\varrho_{23} = 0$) and $B_{213} = -B_{123}$ (that is, $\nabla_1\varrho_{13} = -\nabla_1\varrho_{23}$). Summarizing, the only possibly non-zero components of $\nabla\varrho$ are

$$\nabla_1\varrho_{23} = \nabla_1\varrho_{32} = -\nabla_2\varrho_{13} = -\nabla_2\varrho_{31} = (c-a)\alpha,$$

where $\alpha = B_{123}$ is a constant. In particular, if $\alpha = 0$, then (M,g) is locally symmetric. In the sequel, we then assume $\alpha \neq 0$, and consider the system of partial differential equations

$$e_1\eta = B_{112}, \qquad e_2\eta = B_{212}, \qquad e_3\eta = \frac{a-c}{2\alpha}. \tag{3.13}$$

We can compute $R(e_i,e_j)e_k$ both as a function of the Ricci components and using the covariant derivatives $\nabla_{e_i} e_j$. Comparing the corresponding expressions, standard calculations give

$$\begin{cases} 2\alpha^2 = -a, \\ e_1(B_{212}) - e_2(B_{112}) + B_{112}^2 + B_{212}^2 + a - c = 0, \\ e_3(B_{212}) - \alpha B_{112} = 0. \end{cases} \tag{3.14}$$

Using (3.14), it is easy to check that $[e_i,e_j]\eta = e_i(e_j\eta) - e_j(e_i\eta)$, for all i, j. Hence, a basic theorem on partial differential equations ensures that (3.13) admits a unique solution under an initial condition $\eta_0 = \eta(p)$, with $p \in M$. For such a solution η of (3.13), we can construct a new pseudo-orthonormal frame $\{e_i^*\}$, defined by

$$e_1^* = (\cos\eta)e_1 - (\sin\eta)e_2, \qquad e_2^* = (\sin\eta)e_1 + (\cos\eta)e_2, \qquad e_3^* = e_3, \tag{3.15}$$

and it is easy to check that

$$B^*_{123} = -B^*_{132} = -B^*_{213} = B^*_{231} = \alpha, \qquad B^*_{312} = -B^*_{321} = -\frac{a-c}{2\alpha},$$

and $B^*_{ijk} = 0$ in all the other cases. Therefore, all B^*_{ijk} are constant.

I(c) $a \neq b = c$. We could proceed as in the previous case and show, by direct calculations, that there always exists a pseudo-orthonormal frame field $\{e_i\}$ such that all B_{ijk} are constant. However, it is enough to prove it when

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_2} e_1 = \alpha e_3, & \nabla_{e_3} e_1 = \alpha e_2, \\ \nabla_{e_1} e_2 = 0, & \nabla_{e_2} e_2 = B_{223} e_3, & \nabla_{e_3} e_2 = -\alpha e_1 + B_{323} e_3, \\ \nabla_{e_1} e_3 = 0, & \nabla_{e_2} e_3 = \alpha e_1 + B_{223} e_2, & \nabla_{e_3} e_3 = B_{323} e_3, \end{array} \tag{3.16}$$

where $\alpha = B_{213}$ is a constant and B_{223}, B_{323} are functions.

In fact, in all the other cases, three-dimensional homogeneous Lorentzian manifolds (M, g), having a diagonal Ricci tensor with eigenvalues $q_1 \neq q_2 = q_3$, admit a pseudo-orthonormal frame field $\{e_i\}$ with all B_{ijk} constant [3, pp. 97–100].

We complete this case by proving that even when (3.16) holds, there exists a pseudo-orthonormal frame field $\{e_i^*\}$ such that all B^*_{ijk} are constant. The argument is similar to the one used in the previous case.

We first note that, by (3.9) and (3.16), if $\alpha = 0$, then $\nabla_i \varrho_{jk} = 0$ for all i, j, k, that is, (M, g) is locally symmetric. Therefore, we now assume $\alpha \neq 0$, and consider the system of partial differential equations

$$e_1 \eta = -\frac{b}{2\alpha}, \qquad e_2 \eta = B_{223}, \qquad e_3 \eta = B_{323}. \tag{3.17}$$

We can compute the curvature components both as a function of the Ricci components and starting from (3.16). Comparing the corresponding expressions, we obtain

$$\begin{cases} 2\alpha^2 = -c, \\ e_2(B_{323}) - e_3(B_{223}) - B_{223}^2 + B_{323}^2 + b = 0, \\ e_1(B_{323}) - \alpha B_{223} = e_1(B_{223}) - \alpha B_{323} = 0. \end{cases} \tag{3.18}$$

Because of (3.18), we have $[e_i, e_j]\eta = e_i(e_j \eta) - e_j(e_i \eta)$, for all i, j. Hence, (3.17) admits a unique solution under an initial condition $\eta_0 = \eta(p)$, with $p \in M$. If η is such a solution of (3.17), we put

$$e_1^* = e_1, \qquad e_2^* = (\cosh \eta) e_1 - (\sinh \eta) e_2, \qquad e_3^* = (\sinh \eta) e_1 - (\cosh \eta) e_2. \tag{3.19}$$

Then, one can easily check that $\{e_i^*\}$ is again a pseudo-orthonormal frame (with e_3^* timelike), and

$$B^*_{123} = -B^*_{132} = -\frac{b}{2\alpha}, \qquad B^*_{213} = -B^*_{231} = B^*_{312} = -B^*_{321} = -\alpha,$$

while $B^*_{ijk} = 0$ in all the other cases. Therefore, all B^*_{ijk} are constant.

(II) *Segre type* $\{1z\bar{z}\}$.

In this case, $\varrho_{11} = a$, $\varrho_{22} = \varrho_{33} = b$, $\varrho_{12} = \varrho_{13} = 0$ and $\varrho_{23} = c \neq 0$. Writing (3.5) for $(j, k) = (2, 2)$, $(1, 2)$ and $(1, 3)$, we easily get

$$\begin{cases} \nabla_i \varrho_{22} = -2c B_{i23}, \\ \nabla_i \varrho_{12} = (a - b) B_{i12} - c B_{i13}, \\ \nabla_i \varrho_{13} = -c B_{i12} - (a - b) B_{i13}. \end{cases} \tag{3.20}$$

Since $c \neq 0$ and all $\nabla_i \varrho_{jk}$ are constant, (3.20) implies at once that the B_{ijk} are constant whenever $j \neq k$. This, together with (3.4), implies that all B_{ijk} are constant.

(III) *Segre type* $\{21\}$.

In this case, $\varrho_{11} = a$, $\varrho_{22} = b$, $\varrho_{33} = b + 2\varepsilon$, $\varrho_{12} = \varrho_{13} = 0$ and $\varrho_{23} = 1$, where $\varepsilon = \pm 1$. When $a - b \neq \varepsilon$, we can proceed as in the previous case. In fact, we write (3.5) for $(j, k) = (2, 2)$, $(1, 2)$ and $(1, 3)$ and we get

$$\begin{cases} \nabla_i \varrho_{22} = -\varepsilon B_{i23}, \\ \nabla_i \varrho_{12} = (a - b) B_{i12} - \varepsilon B_{i13}, \\ \nabla_i \varrho_{13} = -\varepsilon B_{i12} - (a - b) B_{i13}. \end{cases} \tag{3.21}$$

If $a - b \neq \varepsilon$, (3.21) implies that the B_{ijk} are constant for $j \neq k$ and so, for all i, j, taking into account (3.4). Then, we are left with the case when $a - b \neq \varepsilon$. Writing (3.5) for all possible j, k, we easily get

$$\begin{cases} \nabla_i \varrho_{11} = 0, \\ \nabla_i \varrho_{22} = \nabla_i \varrho_{33} = \nabla_i \varrho_{23} = -\varepsilon B_{i23}, \\ \nabla_i \varrho_{12} = \nabla_i \varrho_{13} = -\varepsilon (B_{i12} + B_{i13}), \end{cases} \tag{3.22}$$

for all i. In particular,

$$B_{i23} = \text{constant}, \qquad B_{i12} + B_{i13} = \text{constant}, \tag{3.23}$$

for all i. From the divergence formula (3.10), using (3.22) we now get

$$B_{212} + B_{213} = -(B_{312} + B_{313}) = 0, \qquad B_{112} + B_{113} = -\varepsilon (B_{223} + B_{323}). \tag{3.24}$$

Next, we write (3.6) for $(j, k) = (1, 2)$ and $(2, 2)$ and we obtain

$$\nabla^2_{ri} \varrho_{12} = -\varepsilon_i B_{rit} \nabla_t \varrho_{12} - \nabla_i \varrho_{22} (B_{r12} + B_{r13}) - B_{r23} \nabla_i \varrho_{13},$$
$$\nabla^2_{ri} \varrho_{22} = -\varepsilon_i B_{rit} \nabla_t \varrho_{22} + 2 B_{r12} \nabla_i \varrho_{12} - 2 B_{r23} \nabla_i \varrho_{23},$$

that is, taking into account (3.23),

$$\varepsilon_i B_{rit} \nabla_t \varrho_{12} = \text{constant}, \tag{3.25}$$
$$\varepsilon_i B_{rit} \nabla_t \varrho_{22} - 2 B_{r12} \nabla_i \varrho_{12} = \text{constant}. \tag{3.26}$$

We write (3.25) for $(r, i) = (1, 1)$, $(2, 2)$ and $(3, 3)$ and we obtain that $B_{112} \nabla_1 \varrho_{12}$, $B_{212} \nabla_1 \varrho_{12}$ and $B_{313} \nabla_1 \varrho_{12}$ are constant. If $\nabla_1 \varrho_{12} \neq 0$, this implies that B_{112}, B_{212} and B_{313} are constant and so, by (3.23) and (3.4), all B_{ijk} are constant. Hence, we are left with the case when $\nabla_1 \varrho_{12} = 0$, that is, $B_{112} + B_{113} = 0$.

From (3.26), for $i = 1$ and taking into account $\nabla_1 \varrho_{12} = 0$, we now have

$$B_{r12} \nabla_2 \varrho_{22} + B_{r13} \nabla_3 \varrho_{22} = \text{constant},$$

that is, by (3.22),

$$\varepsilon B_{223} (B_{r13} - B_{r12}) = \text{constant}. \tag{3.27}$$

If $B_{223} \neq 0$, (3.27) and (3.23) imply at once that B_{r12}, B_{r13} are constant for all r. So, in the sequel we also assume $B_{223} = 0$. From the second equation in (3.24) we get $B_{323} = 0$ (since $B_{112} + B_{113} = 0$). So, (3.22) yields $\nabla_2 \varrho_{ij} = \nabla_3 \varrho_{ij} = 0$ for all i, j. Finally, using this information in (3.26) for $i = 2$, we now get

$$-B_{r12} \nabla_1 \varrho_{22} = \varepsilon_i B_{r2t} \nabla_t \varrho_{22} - 2 B_{r12} \nabla_2 \varrho_{12} = \text{constant}. \tag{3.28}$$

If $\nabla_1 \varrho_{22} \neq 0$, then (3.28) implies that the B_{r12} are constant and, by (3.23), also the B_{r13} are constant and the conclusion follows. On the other hand, if $\nabla_1 \varrho_{22} = 0$, then $\nabla_i \varrho_{jk} = 0$ for all i, j, k, that is, (M, g) is symmetric.

(IV) *Segre type* $\{3\}$. In this case, $\varrho_{11} = \varrho_{22} = \varrho_{33} = b$, $\varrho_{12} = -\varrho_{13} = a \neq 0$ and $\varrho_{23} = 0$. We now write (3.5) for $(j, k) = (1, 2)$, $(2, 2)$ and $(3, 3)$ and we get

$$\begin{cases} \nabla_i \varrho_{12} = a B_{i23}, \\ \nabla_i \varrho_{22} = 2a B_{i12}, \\ \nabla_i \varrho_{33} = 2a B_{i13}. \end{cases} \tag{3.29}$$

Therefore, taking into account (3.3) and (3.4), the B_{ijk} are constant for all j, k and this ends the proof. □

Remark. A three-dimensional Lorentzian manifold (M, g) is locally homogeneous if and only if it is curvature homogeneous up to order two [3, Remarks 1,2,3]. From Theorem 1.1, we then get at once the following

Theorem 3.1. *Let (M, g) be a three-dimensional Lorentzian manifold. The following conditions are equivalent:*

(i) *(M, g) is curvature homogeneous up to order two;*
(ii) *(M, g) is locally homogeneous;*
(iii) *(M, g) is either locally symmetric or locally isometric to a Lie group equipped with a left-invariant Lorentzian metric.*

4. The classification of three-dimensional homogeneous Lorentzian manifolds

Rahmani [15] classified three-dimensional unimodular Lie groups equipped with a left-invariant Lorentzian metric, obtaining a result corresponding to the one found by Milnor [11] in the Riemannian case. Earlier, Cordero and Parker [8] already studied three-dimensional Lie groups equipped with left-invariant Lorentzian metrics, determining their curvature tensors and investigating the symmetry groups of the sectional curvature in the different cases. In particular, they wrote down the possible forms of a non-unimodular Lie algebra. Taking into account these results and Theorem 1.1, we obtain the following:

Theorem 4.1. *Let (M, g) be a three-dimensional connected, simply connected, complete homogeneous Lorentzian manifold. If (M, g) is not symmetric, then $M = G$ is a three-dimensional Lie group and g is left-invariant. Precisely, one of the following cases occurs:*

- *If G is unimodular, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with e_3 timelike, such that the Lie algebra of G is one of the following:*

(a)

$$(\mathfrak{g}_1):\quad \begin{aligned} &[e_1, e_2] = \alpha e_1 - \beta e_3,\\ &[e_1, e_3] = -\alpha e_1 - \beta e_2,\\ &[e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3 \quad \alpha \neq 0. \end{aligned} \tag{4.1}$$

In this case, $G = O(1, 2)$ or $SL(2, \mathbb{R})$ if $\beta \neq 0$, while $G = E(1, 1)$ if $\beta = 0$.

(b)

$$(\mathfrak{g}_2):\quad \begin{aligned} &[e_1, e_2] = \gamma e_2 - \beta e_3,\\ &[e_1, e_3] = -\beta e_2 + \gamma e_3, \quad \gamma \neq 0,\\ &[e_2, e_3] = \alpha e_1. \end{aligned} \tag{4.2}$$

In this case, $G = O(1, 2)$ or $SL(2, \mathbb{R})$ if $\alpha \neq 0$, while $G = E(1, 1)$ if $\alpha = 0$.

(c)

$$(\mathfrak{g}_3):\quad \begin{aligned} &[e_1, e_2] = -\gamma e_3,\\ &[e_1, e_3] = -\beta e_2,\\ &[e_2, e_3] = \alpha e_1. \end{aligned} \tag{4.3}$$

Table 1 lists all the Lie groups G which admit a Lie algebra g_3, taking into account the different possibilities for α, β and γ.

Table 1
Unimodular Lie groups with Lie algebra g_3

G	α	β	γ
$O(1, 2)$ or $SL(2, \mathbb{R})$	+	+	+
$O(1, 2)$ or $SL(2, \mathbb{R})$	+	−	−
$SO(3)$ or $SU(2)$	+	+	−
$E(2)$	+	+	0
$E(2)$	+	0	−
$E(1, 1)$	+	−	0
$E(1, 1)$	+	0	+
H_3	+	0	0
H_3	0	0	−
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	0	0	0

(d)

$$(\mathfrak{g}_4):\quad \begin{aligned} &[e_1, e_2] = -e_2 + (2\varepsilon - \beta) e_3, \quad \varepsilon = \pm 1,\\ &[e_1, e_3] = -\beta e_2 + e_3,\\ &[e_2, e_3] = \alpha e_1. \end{aligned} \tag{4.4}$$

Table 2 describes all Lie groups G admitting a Lie algebra g_4.

Table 2
Unimodular Lie groups with Lie algebra g_4

G ($\varepsilon = 1$)	α	β	G ($\varepsilon = -1$)	α	β
$O(1,2)$ or $SL(2,\mathbb{R})$	$\neq 0$	$\neq 1$	$O(1,2)$or $SL(2,\mathbb{R})$	$\neq 0$	$\neq -1$
$E(1,1)$	0	$\neq 1$	$E(1,1)$	0	$\neq -1$
$E(1,1)$	<0	1	$E(1,1)$	>0	-1
$E(2)$	>0	1	$E(2)$	<0	-1
H_3	0	1	H_3	0	-1

- *If G is non-unimodular, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with e_3 timelike, such that the Lie algebra of G is one of the following:*

(e)

$$(\mathfrak{g}_5): \quad \begin{aligned} &[e_1, e_2] = 0, \\ &[e_1, e_3] = \alpha e_1 + \beta e_2, \\ &[e_2, e_3] = \gamma e_1 + \delta e_2, \qquad \alpha + \delta \neq 0, \qquad \alpha\gamma + \beta\delta = 0. \end{aligned} \tag{4.5}$$

(f)

$$(\mathfrak{g}_6): \quad \begin{aligned} &[e_1, e_2] = \alpha e_2 + \beta e_3, \\ &[e_1, e_3] = \gamma e_2 + \delta e_3, \qquad \alpha + \delta \neq 0, \qquad \alpha\gamma - \beta\delta = 0, \\ &[e_2, e_3] = 0. \end{aligned} \tag{4.6}$$

(g)

$$(\mathfrak{g}_7): \quad \begin{aligned} &[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, \\ &[e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, \\ &[e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \qquad \alpha + \delta \neq 0, \qquad \alpha\gamma = 0. \end{aligned} \tag{4.7}$$

Proof. By Theorem 1.1, if M is not symmetric, then it is isometric to a three-dimensional Lie group G equipped with a left-invariant Lorentz metric. Assume first that there exists a linear mapping from $\mathfrak{g}$ to $\mathbb{R}$, such that

$$[x, y] = l(x)y - l(y)x, \tag{4.8}$$

for all $x, y \in \mathfrak{g}$. Then, any Lorentzian metric on G has constant sectional curvature, and this constant can be any real number [13, Theorem 1]. In particular, G is symmetric. So, in the sequel we shall assume G does not satisfy (4.8).

In [15], Rahmani introduced a cross-product $X \times Y$ adapted to the Lorentzian environment, and the four possibilities Rahmani found for the unimodular Lie algebras ($\mathfrak{g}_1$)–($\mathfrak{g}_4$), correspond to the four possible forms of the self-adjoint transformation L, defined by

$$[X, Y] = L(X \times Y).$$

Following [8], cases ($\mathfrak{g}_5$)–($\mathfrak{g}_7$) are the possible forms of the non-unimodular Lie algebra of a three-dimensional Lorentzian Lie group, rewritten here for a Lorentzian metric of signature $(+,+,-)$ and a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ with e_3 timelike. The determinant $D = \frac{4(\alpha\delta - \beta\gamma)}{(\alpha+\delta)^2}$ provides a complete isomorphism invariant for Lie algebras ($\mathfrak{g}_5$)–($\mathfrak{g}_7$). □

5. Three-dimensional Lorentzian symmetric spaces

We can now complete the classification of three-dimensional homogeneous Lorentzian manifolds, by classifying the symmetric ones.

Let (M, g) be a three-dimensional Lorentzian symmetric space. We can consider separately two cases.

(A) (M, g) *is not isometric to a three-dimensional Lie group*.

Following the proof of Theorem 1.1, we see that this can only happen for some of the possible forms of the Ricci operator. More precisely, one of the following cases must occur.

(A1) The Ricci operator of (M, g) is diagonal with eigenvalues $q_1 = q_2 = q_3$.

Therefore, (M, g) is a three-dimensional Einstein space and so, it has constant sectional curvature. If M is connected and simply connected, then (M, g) is one of the *Lorentzian space forms* S^3_1, $\mathbb{R}^3_1$ or $\mathbb{H}^3_1$, of positive, null and negative constant sectional curvature, respectively [12].

(A2) The Ricci operator of (M, g) is diagonal and has eigenvalues $q_1 = q_2 \neq q_3$, and $B_{ijk} = 0$ for all $(i, j, k) \neq (1, 1, 2)$ or $(2, 1, 2)$.
Note that $\nabla_{e_i} e_3 = 0$ for all i. Therefore, e_3 is a timelike parallel vector field and so, M is reducible as a direct product $M^2 \times \mathbb{R}$, where M^2 is a Riemannian surface. Since M is symmetric, M^2 itself is symmetric and so, it has constant sectional curvature. If M is connected and simply connected, (M, g) is then isometric to either $S^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$.
(A3) The Ricci operator of (M, g) is diagonal and has eigenvalues $q_1 \neq q_2 = q_3$, and $B_{ijk} = 0$ for all $(i, j, k) \neq (2, 2, 3)$ or $(3, 2, 3)$.

This case is very similar to the previous one. In fact, $\nabla_{e_i} e_1 = 0$ for all i. Thus, e_1 is a spacelike parallel vector field and M is reducible as a direct product $\mathbb{R} \times M_1^2$, where M_1^2 is a Lorentzian surface. Since M is symmetric, also M_1^2 is symmetric and so, it has constant sectional curvature. When M is connected and simply connected, (M, g) is isometric to either $\mathbb{R} \times S_1^2$ or $\mathbb{R} \times \mathbb{H}_1^2$.
(A4) The Ricci operator of (M, g) is of Segre type {21} with $a - b = \varepsilon$, and

$$\begin{aligned} &\nabla_{e_1} e_1 = B_{112} e_2 - B_{112} e_3, && \nabla_{e_2} e_1 = B_{212} e_2 - B_{212} e_3, && \nabla_{e_3} e_1 = B_{312} e_2 - B_{312} e_3, \\ &\nabla_{e_1} e_2 = -B_{112} e_1, && \nabla_{e_2} e_2 = -B_{212} e_1, && \nabla_{e_3} e_2 = -B_{312} e_1, \\ &\nabla_{e_1} e_3 = -B_{112}, && \nabla_{e_2} e_3 = -B_{212} e_1, && \nabla_{e_3} e_3 = -B_{312} e_1. \end{aligned} \tag{5.1}$$

Put $u = e_2 - e_3$. Then, $\nabla_{e_i} u = 0$ for all i, that is, u is a parallel null vector field. Three-dimensional symmetric spaces admitting a parallel null vector field were completely classified in [7]. A three-dimensional locally symmetric Lorentzian manifold (M, g), having a parallel null vector field, admits local coordinates (t, x, y) such that, with respect to the local frame field $\{(\frac{\partial}{\partial t}), (\frac{\partial}{\partial x}), (\frac{\partial}{\partial y})\}$, the Lorentzian metric g and the Ricci operator are given by

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & -\frac{1}{\varepsilon}\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.2}$$

where $\varepsilon = \pm 1$, $u = (\frac{\partial}{\partial t})$ and

$$f(x, y) = x^2 \alpha + x\beta(y) + \xi(y), \tag{5.3}$$

for any constant $\alpha \in \mathbb{R}$ and any functions β, ξ [7, Theorem 6]. It is easy to build a (local) pseudo-orthonormal frame field from $\{(\frac{\partial}{\partial t}), (\frac{\partial}{\partial x}), (\frac{\partial}{\partial y})\}$, and to check that, whenever $\alpha f \neq 0$ (that is, g is not flat), the Ricci operator described by (5.2) is of Segre type {21}.
(B) $(M = G, g)$ *is a three-dimensional Lie group*.

In this case, M is one of the Lie groups listed in Theorem 4.1. The case described by (4.8) is trivial, since all Lorentzian metrics have constant sectional curvature. For all the remaining cases $(\mathfrak{g}_1)$–$(\mathfrak{g}_7)$, we can determine the covariant derivatives using (3.8). Then, we can compute the curvature components and, by (3.5) and (3.6), the components of ϱ and $\nabla\varrho$. Obviously, the symmetric cases are exactly the ones for which all components of $\nabla\varrho$ vanish.

All these calculations are long but very standard, and will be inserted in detail in a forthcoming paper [5], in which we shall also consider Einstein-like Lorentzian metrics on three-dimensional homogeneous Lorentzian manifolds. We report here the conclusions we obtain in the different cases.
(B1) A three-dimensional Lorentzian Lie group G, having either $\mathfrak{g}_1$ or $\mathfrak{g}_2$ as Lie algebra, is never symmetric. In fact, in the case of $\mathfrak{g}_1$, we easily get that G is symmetric if and only if $\alpha = 0$, which contradicts (4.1). In a similar way, in the case of $\mathfrak{g}_2$, we get $\gamma = 0$, against (4.2).
(B2) When (G, g) has a Lie algebra of diagonal type $\mathfrak{g}_3$, G is symmetric if and only if

$$\begin{cases} (q_2 - q_3)(\alpha - \beta - \gamma) = 0, \\ (q_1 - q_3)(\alpha - \beta + \gamma) = 0, \\ (q_1 - q_2)(\alpha + \beta - \gamma) = 0, \end{cases} \tag{5.4}$$

where q_1, q_2, q_3 are the Ricci eigenvalues (which depend on α, β, γ). Taking into account Table 1, we obtain that (5.4) holds if and only if one of the following cases occurs:

- $\alpha = \beta = \gamma$. If $\alpha \neq 0$, then $G = O(1,2)$ or $SL(2,\mathbb{R})$ and g has negative constant sectional curvature $-\frac{\alpha^2}{4}$. If $\alpha = 0$, we find $G = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ (and g is flat).
- $\alpha - \gamma = \beta = 0$. In this case, $G = E(1,1)$ and g is flat.
- $\alpha - \beta = \gamma = 0$. Then, $G = E(2)$ and g is flat.

(B3) When (G, g) has a Lie algebra of type $\mathfrak{g}_4$, it is symmetric if and only if

$$\begin{cases} (\alpha + 2\varepsilon - 2\beta)^2 = 0, \\ \alpha(4\beta - 4\varepsilon - 3\alpha) = 0, \\ \alpha(\alpha^2 - \alpha\beta - 2\varepsilon\alpha + 4\varepsilon\beta - 4) = 0, \\ \alpha(\alpha^2 - \alpha\beta + 4\varepsilon\alpha - 4\varepsilon\beta + 4) = 0. \end{cases} \tag{5.5}$$

From (5.5) it follows that G is symmetric if and only if $\alpha = \beta - \varepsilon = 0$. In this case, taking into account Table 2, we have $G = H_3$. Moreover, g is flat.

(B4) A Lie group (G, g), having a Lie algebra of type $\mathfrak{g}_5$, is symmetric if and only if

$$\begin{cases} \alpha\gamma + \beta\delta = 0 \\ \alpha(\delta^2 - \alpha\delta + \beta\gamma + \gamma^2) = 0, \\ (\beta + \gamma)(\alpha^2 - \alpha\delta + \beta^2 + \beta\gamma) = 0, \\ (\beta + \gamma)(\delta^2 - \alpha\delta + \beta\gamma + \gamma^2) = 0, \\ \delta(\alpha^2 - \alpha\delta + \beta^2 + \beta\gamma) = 0, \\ (\beta - \gamma)\beta\delta = 0, \\ (\beta - \gamma)(\gamma^2 - \beta^2 + \delta^2 - \alpha^2) = 0. \end{cases} \tag{5.6}$$

(The first equation of (5.6) comes from (4.5).) Taking into account $\alpha + \delta \neq 0$, (5.6) is satisfied if and only if one of the following cases occurs:

- $\beta = \gamma = \delta = 0$ and $\alpha \neq 0$. In this case, e_2 is a spacelike parallel vector field and so, G is reducible. Since the Ricci eigenvalues are $q_1 = q_3 = \alpha^2$ and $q_2 = 0$, G is locally isometric to $\mathbb{R} \times S^2_1$.
- $\alpha = \beta = \gamma = 0$ and $\delta \neq 0$. In this case, e_1 is a spacelike parallel vector field, G is reducible and the Ricci eigenvalues are $q_1 = 0$ and $q_2 = q_3 = 0$. So, G is locally isometric to $\mathbb{R} \times S^2_1$.
- $\beta + \gamma = 0$ and $\delta = \alpha \neq 0$. In this case, $q_1 = q_2 = q_3 = 2\alpha^2$. Hence, G has constant sectional curvature $\alpha^2 > 0$.

(B5) A Lie group (G, g), having a Lie algebra of type $\mathfrak{g}_6$, is symmetric if and only if

$$\begin{cases} \alpha\gamma - \beta\delta = 0, \\ (\beta + \gamma)(\delta^2 - \alpha^2 + \beta^2 - \gamma^2) = 0, \\ \alpha(\delta^2 - \alpha\delta + \beta\gamma - \gamma^2) = 0, \\ (\beta - \gamma)(\alpha\delta - \alpha^2 + \beta^2 - \beta\gamma) = 0, \\ (\beta - \gamma)(\delta^2 - \alpha\delta + \beta\gamma - \gamma^2) = 0, \\ \delta(\alpha\delta - \alpha^2 + \beta^2 - \beta\gamma) = 0. \end{cases} \tag{5.7}$$

Standard calculations show that (5.7) holds if and only if one of the following cases occurs:

- $\alpha = \beta = \gamma = 0$ and $\delta \neq 0$. In this case, G is reducible (e_2 is a spacelike parallel vector field) and the Ricci eigenvalues are $q_1 = q_3 = -\delta^2 < 0$ and $q_2 = 0$. Hence, G is locally isometric to $\mathbb{R} \times \mathbb{H}^2_1$.
- $\beta = \gamma = \delta = 0$ and $\alpha \neq 0$. Then, G is reducible, because e_3 is a timelike parallel vector field. Since the Ricci eigenvalues are $q_1 = q_2 = -\alpha^2 < 0$ and $q_3 = 0$, G is locally isometric to $\mathbb{H}^2 \times \mathbb{R}$.
- $\gamma = \beta$ and $\delta = \alpha \neq 0$. In this case, the Ricci tensor is diagonal and $q_1 = q_2 = q_3 = -2\alpha^2$. Hence, G has constant sectional curvature $-\alpha^2 < 0$.
- $\beta = \varepsilon\alpha$ and $\gamma = \varepsilon\delta$, with $\alpha + \delta \neq 0$ and $\varepsilon = \pm 1$. In this case, the Ricci tensor is diagonal and $q_1 = q_2 = q_3 = -\frac{(\alpha+\delta)^2}{2}$. Hence, G has constant sectional curvature $-\frac{(\alpha+\delta)^2}{4} < 0$.

(B6) When a Lorentzian Lie group (G, g) has a Lie algebra of type $\mathfrak{g}_7$, it is symmetric if and only if

$$\begin{cases} \alpha\gamma = 0, \\ \beta\gamma^2 = 0, \\ \delta(\alpha^2 - \alpha\delta + \beta\gamma) = 0, \\ \gamma^2(\gamma - 3\beta) = 0, \\ \gamma^2(\gamma + 3\beta) = 0, \end{cases} \tag{5.8}$$

Since $\alpha + \delta \neq 0$, (5.8) is satisfied if and only if one of the following cases occurs:

- $\alpha = \gamma = 0$ and $\delta \neq 0$. In this case, $q_1 = q_2 = q_3 = 0$, that is, G is flat.
- either $\gamma = \delta = 0$ and $\alpha \neq 0$, or $\gamma = 0$ and $\alpha = \delta \neq 0$. The Ricci components are $\varrho_{11} = 0$, $\varrho_{22} = -\varrho_{33} = -\alpha^2$, $\varrho_{12} = \varrho_{13} = 0$ and $\varrho_{23} = \alpha^2$, that is, the Ricci tensor is of Segre type $\{21\}$. Moreover, $u = e_2 + e_3$ is a parallel null vector field. Therefore, g is described by (5.2) and (5.3).

Therefore, we proved the following

Theorem 5.1. *A connected, simply connected three-dimensional Lorentzian symmetric space (M, g) is either*

(i) *a Lorentzian space form S^3_1, $\mathbb{R}^3_1$ or $\mathbb{H}^3_1$, or*
(ii) *a direct product $\mathbb{R} \times S^2_1$, $\mathbb{R} \times \mathbb{H}^2_1$, $S^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, or*
(iii) *a space with a Lorentzian metric g described by* (5.2) *and* (5.3).

Three-dimensional unimodular Lie groups, having a flat Lorentzian metric, have been classified in [13] (see also [15]). Our analysis above leads to the following extension:

Theorem 5.2. *Let (G, g) be a three-dimensional Lorentzian Lie group and $\mathfrak{g}$ its Lie algebra. Apart from the trivial case described by* (4.8), *we have:*

(a) *(G, g) is flat if and only if*
- *$\mathfrak{g} = \mathfrak{g}_3$ and either $G = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ with $\alpha = \beta = \gamma = 0$, $G = E(1, 1)$ with $\alpha - \gamma = \beta = 0$, or $G = E(2)$ with $\alpha - \beta = \gamma = 0$, or*
- *$\mathfrak{g} = \mathfrak{g}_4$ and $G = H_3$ with $\alpha = \beta - \varepsilon = 0$, or*
- *$\mathfrak{g} = \mathfrak{g}_7$ with either $\alpha = \gamma = 0$ and $\delta \neq 0$, or $\gamma = 0$ and $\alpha = \delta \neq 0$.*

(b) *(G, g) has positive constant sectional curvature if and only if $\mathfrak{g} = \mathfrak{g}_5$ with $\beta + \gamma = 0$ and $\delta = \alpha \neq 0$.*
(c) *(G, g) has negative constant sectional curvature if and only if*
- *$\mathfrak{g} = \mathfrak{g}_3$ and $G = O(1, 2)$ or $SL(2, \mathbb{R})$ with $\alpha = \beta = \gamma \neq 0$, or*
- *$\mathfrak{g} = \mathfrak{g}_6$ with either $\gamma = \beta$ and $\delta = \alpha \neq 0$, or $\beta = \varepsilon\alpha$ and $\gamma = \varepsilon\delta$ (with $\alpha + \delta \neq 0$ and $\varepsilon = \pm 1$).*

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